

# The Mass Effect in Quarkonium Decay

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## Abstract

In the NRQCD factorization for productions and decays of a quarkonium the difference between the quarkonium mass and the twice of the heavy quark mass is neglected at the leading order of the small velocity expansion. The effect of this difference is included in the relativistic correction. We attempt to define the difference in terms of NRQCD matrix elements and to separate the effect of the difference in several decays. It turns out that the total relativistic correction can be estimated by the difference. Numerically the correction will enhance the decay widths considered here.

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Quarkonium systems generally are thought as simpler than light hadrons and it may be easier to handle them in the framework of QCD. However, only recently we have been able to treat their decays and productions rigorously, based on a factorization with non-relativistic QCD(NRQCD) [1], where the effect of short distance is handled with perturbative QCD and the effect of long distance is parameterized with NRQCD matrix elements. The factorization is performed by utilizing the fact that a heavy quark moves with a small velocity  $v$  in the quarkonium rest frame and an expansion in  $v$  can be employed. In this factorization, a quarkonium decay, e.g., like  $J/\Psi$ , can be imagined at the leading order of  $v$  as the following: The  $c$ - and  $\bar{c}$ - quark in  $J/\Psi$  has certain probability to be freed at the same space point and this  $c\bar{c}$  pair decays subsequently. The probability has a nonperturbative nature and is at order of  $v^0$ , while the decay of the  $c\bar{c}$  pair can be treated with perturbative QCD. Various effects at higher order of  $v$  can be taken into account, e.g., relativistic effect, and the effect of that the freed  $c\bar{c}$  pair does not possess the same quantum numbers as those of  $J/\Psi$ . In order to make the factorization consistently the quarkonium mass is forced to be approximated as twice of the heavy quark mass. In general the quarkonium mass is not the same as twice of the heavy quark mass, the difference, we call it as the mass difference, is at order of  $v^2$  and is made by nonperturbative physics, which binds the heavy-, antiheavy quark and other possible light dynamical freedoms into a bound state. This difference makes the phase space of a quarkonium decay differently than that of a heavy quark pair decay and introduces a correction at order of  $v^2$ . The situation here is relatively similar as this in deeply inelastic scattering, where the mass of the initial hadron introduces the target mass effect [2]. This effect can be thought as a kinematical effect at higher-twist, the other higher twist effects are dynamical, which come from the fact that partons inside a target can have a transverse momentum and can be off-shell,  $\dots$  etc., these together are corrections to the parton model. In a quarkonium decay, like  $J/\Psi$ -decay, the correction from the next-to-leading order of  $v$  is characterized by a single matrix element in NRQCD and both kinematic- and dynamical effects are included in the correction, as shown below explicitly in considered processes. However, the kinematic effect can be separated from the high-order corrections, and it can be interpreted in term of the mass difference. A decay width obtained with and without the separation should be at the same accuracy at the considered orders. This fact leads to that the effect at the next-leading order of  $v^2$  can be interpreted totally by the mass difference in the decays considered here. In this work we will first try to define the mass difference in terms of NRQCD matrix elements and then calculate the correction from the next-to-leading order with and without the separation.

We start with the energy momentum tensor  $T_{\mu\nu}$  of QCD, a nice discussion about properties of the tensor and interesting references can be found in [3]. For our purpose we consider the full QCD containing gluons,  $N_f$  flavors of light quarks and one flavor of heavy quark  $Q$ . The tensor then can be written as

$$T^{\mu\nu}(x) = T_Q^{\mu\nu}(x) + T_G^{\mu\nu}(x) + T_q^{\mu\nu}(x) \quad (1)$$

where  $T_Q^{\mu\nu}$  is the contribution from the heavy quark,  $T_G^{\mu\nu}$  is from gluons and  $T_q^{\mu\nu}$  is from light flavors. For a state of a hadron with momentum  $p$  one has by definition:

$$\frac{\langle p | \int d^3x T^{0\mu}(x) | p \rangle}{\langle p | p \rangle} = p^\mu. \quad (2)$$

If the hadron is at rest, the zeroth component in the right side of Eq.(2) is the mass  $M$  of the hadron. The energy momentum tensor can be decomposed into a trace part and a traceless part,

$$T^{\mu\nu}(x) = \bar{T}^{\mu\nu}(x) + \hat{T}^{\mu\nu}(x), \quad (3)$$

$$g_{\mu\nu}\bar{T}^{\mu\nu}(x) = 0. \quad (4)$$

From the space-time covariance one can derive

$$p^0\langle p|T_{\mu\nu}|p\rangle = p_\mu p_\nu, \quad (5)$$

$$p^0\langle p|\bar{T}_{\mu\nu}|p\rangle = p_\mu p_\nu - \frac{1}{4}g_{\mu\nu}M^2, \quad (6)$$

$$p^0\langle p|\hat{T}_{\mu\nu}|p\rangle = \frac{1}{4}g_{\mu\nu}M^2 \quad (7)$$

where the factor  $p^0$  appears because we take the normalization of a state as  $\langle p|p'\rangle = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')$ . To interpret the mass difference we match the heavy quark fields of full QCD into NRQCD fields. We perform this matching at the tree-level and for consistency we neglect the contributions from the trace-anomaly and the anomalous dimension of quark masses. The heavy quark contribution  $T_Q^{\mu\nu}$  in the full QCD reads:

$$T_Q^{\mu\nu} = \bar{T}_Q^{\mu\nu} + \hat{T}_Q^{\mu\nu}, \quad (8)$$

$$\bar{T}_Q^{\mu\nu} = \frac{1}{2}\{\bar{Q}(iD^\mu\gamma^\nu + iD^\nu\gamma^\mu)Q - \frac{1}{2}g^{\mu\nu}\bar{Q}i\gamma\cdot DQ\}, \quad (9)$$

$$\hat{T}_Q^{\mu\nu} = \frac{1}{4}g^{\mu\nu}m_Q\bar{Q}Q. \quad (10)$$

In the above  $Q$  is the dirac field for the heavy quark  $Q$ ,  $D^\mu$  is the covariant derivative. The last term with  $g^{\mu\nu}$  in  $\bar{T}_Q^{\mu\nu}$  can be simplified with the equation of motion. We take the state  $|p\rangle$  as a quarkonium state  $|H\rangle$  and match the matrix element of  $\hat{T}_Q^{\mu\nu}$  into NRQCD matrix elements, we obtain:

$$M_H = 2m_Q - \langle H|T_K|H\rangle + \sum_f m_f\langle H|\bar{q}_f q_f|H\rangle + O(v^4) + O(\alpha_s) \quad (11)$$

$$T_K = -\frac{1}{2m_Q}\psi_Q^\dagger\mathbf{D}^2\psi_Q + \frac{1}{2m_Q}\chi_Q^\dagger\mathbf{D}^2\chi_Q, \quad (12)$$

where  $T_K$  is an operator defined in NRQCD,  $\psi_Q$  and  $\chi_Q$  are the quark and antiquark fields in NRQCD respectively,  $M_Q$  is the pole mass of the heavy quark. The operator  $T_K$  measures the sum of kinetic energies of a heavy and antiheavy quark. For  $H$  with a nonzero spin the average of the spin is implied. The summation is over all light flavors. The matrix elements  $\langle H|\bar{q}_f q_f|H\rangle$  are at least at the order of  $v^2$ . This can be estimated by taking a  $Q\bar{Q}$  pair instead of the state  $H$ , because the operator  $\bar{q}_f q_f$  is a color-singlet, the matrix element is nonzero provided that at least two gluons must be exchanged between the operator and the  $Q\bar{Q}$  state. A gluon coupled to  $Q$  or  $\bar{Q}$  gives a factor  $v$ . The contribution from this operator to  $M_H$  is suppressed by a factor  $m_f/m_Q$ , relatively to the contribution from  $T_K$ , can be neglected

for charmonium as the mass ratio is small enough. For bottomonium system, because charm quarks are included in the full QCD, neglecting this contribution from charm quark may be problematic, because the mass ratio is not so small. However the matrix element  $\langle H|\bar{c}c|H\rangle$ , where  $c$  and  $\bar{c}$  is the dirac field for charm quark, is suppressed by  $v^2\Lambda_{\text{QCD}}^2/m_c^2$  at least, hence the contribution from this operator suppressed by  $\Lambda_{\text{QCD}}^2/m_b m_c$  relatively to that from  $T_K$ . Giving the fact that  $\Lambda_{\text{QCD}}$  as a characteristic scale for long-distance physics is at order of several hundred MeV, this contribution can be neglected too.

Neglecting the contributions proportional to  $m_f$  and other higher order effects we obtain

$$M_H = 2m_Q - \langle H|T_K|H\rangle. \quad (13)$$

Taking this result we obtain another relation in the quarkonium rest frame within our approximation:

$$2\langle H|T_K|H\rangle = -\langle H|T_G^{00} + T_q^{00}|H\rangle. \quad (14)$$

This relation may be regarded as a field theory version of the virial theorem of quantum mechanics, which relates the expectation value of the kinetic energy to the expectation value of the potential energy. The above equations hold only at the tree-level approximation in field theory. If one takes the contributions from the trace-anomaly, the contributions from the anomalous dimension of quark masses and those from the matching beyond the tree-level, into account, these relations become more complicated. For higher excited states of quarkonia the relation needs to be corrected by higher orders of  $v^2$  as  $v^2$  becomes larger for these states. The matrix element  $\langle H|T_K|H\rangle$  is positive and it indicates that  $M_H < 2m_Q$ .

If the mass of a quarkonium and the pole mass of the heavy quark are known, one may determines how large the matrix element  $\langle H|T_K|H\rangle$  or the binding energy is. We take  $\Upsilon$  and  $J/\Psi$  as examples. The pole mass of b-quark is determined precisely from a study of lattice QCD [5], whose result is  $m_b = 5.0\text{GeV}$ . For charm quark we take the pole mass determined from the D meson semileptonic decay [4], the value is  $m_c = 1.65\text{GeV}$ . We obtain:

$$\langle \Upsilon|T_K|\Upsilon\rangle \approx 0.54\text{GeV}, \quad (15)$$

$$\langle J/\Psi|T_K|J/\Psi\rangle \approx 0.20\text{GeV}. \quad (16)$$

With these values one can also determine the velocity of b- or c-quark in  $\Upsilon$  or  $J/\Psi$  respectively, by identifying  $\langle H|T_K|H\rangle = m_Q v^2$ . We obtain  $v^2 \approx 0.1$  for b-quark and  $v^2 \approx 0.12$  for c-quark. The velocity for c-quark is smaller than that from usual estimation. For c-quark it should be noted that a precise determination of the pole mass is still lacking and the higher order effect in Eq.(13) may be significant.

Now we consider the decay  $J/\Psi \rightarrow \ell^+ \ell^-$ . Starting with the relevant  $S$ -operator we obtain the  $S$ -matrix element

$$\begin{aligned} \langle \ell^+(k_1), \ell^-(k_2)|S|J/\Psi(P)\rangle &= -ie^2 Q_c (2\pi)^4 \delta^4(P - k_1 - k_2) \\ &\quad \cdot \int \frac{d^4 q}{(2\pi)^4} \bar{u}(k_2) \gamma_\mu v(k_2) \frac{1}{q^2} \text{Tr}(\gamma^\mu \Gamma(q, P)) \\ \Gamma_{ij}(q, P) &= \int d^4 x e^{iq \cdot x} \langle 0|\bar{c}(x)_j c(x)_i|J/\Psi\rangle \end{aligned} \quad (17)$$

To perform the NRQCD factorization we match the nonperturbative object  $\Gamma(q, P)$  into NRQCD matrix elements by the small velocity expansion. Up to the order of  $v^2$  it reads:

$$\begin{aligned}
\Gamma_{ij}(q, P) = & (2\pi)^4 \delta^4(q - p_0) \left[ -\frac{1}{2} (P_+ \gamma^l P_-)_{ij} \langle 0 | \chi_c^\dagger \sigma^l \psi_c | J/\Psi \rangle \right. \\
& + \frac{1}{2} (P_- \gamma^l P_+)_{ij} \frac{1}{12m_c^2} \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma^l \psi_c | J/\Psi \rangle \\
& - \frac{1}{2} (P_+ \gamma^l P_-)_{ij} \frac{1}{4m_c^2} \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma^l \psi_c | J/\Psi \rangle \Big] \\
& - \frac{\partial}{\partial q^0} (2\pi)^4 \delta^4(q - p_0) (P_+ \gamma^l P_-)_{ij} \frac{1}{m_c} \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma^l \psi_c | J/\Psi \rangle
\end{aligned} \tag{18}$$

where  $p_0^\mu = (2m_c, 0, 0, 0)$ ,  $P_\pm = (1 \pm \gamma^0)/2$ . The first term is at order of  $v^0$ , the remaining terms are at order of  $v^2$ . The relativistic correction is not only from these terms but also from  $P^\mu = (M_{J/\Psi}, 0, 0, 0)$  in the  $S$ -matrix element. For the terms at  $v^2$  in  $\Gamma(q, P)$  it can be replaced by  $P = p_0$  because we neglect the effect at higher orders. For the first term in  $\Gamma(q, P)$  we can write the  $S$ -matrix element with translational invariance as:

$$\begin{aligned}
\langle \ell^+(k_1), \ell^-(k_2) | S | J/\Psi(P) \rangle = & ie^2 Q_c \frac{1}{4m_c^2} \bar{u}(k_2) \gamma^i v(k_2) (2\pi)^4 \delta^4(P - k_1 - k_2) \langle 0 | \chi_c^\dagger \sigma^i \psi_c | J/\Psi \rangle \\
& + \dots \\
= & ie^2 Q_c \frac{1}{4m_c^2} \bar{u}(k_2) \gamma^i v(k_2) \int d^4x e^{ix \cdot (k_1 + k_2 - P)} \langle 0 | \chi_c^\dagger \sigma^i \psi_c | J/\Psi \rangle \\
& + \dots \\
= & ie^2 Q_c \frac{1}{4m_c^2} \bar{u}(k_2) \gamma^i v(k_2) \int d^4x e^{ix \cdot (k_1 + k_2 - p_0)} \langle 0 | \chi_c^\dagger(x) \sigma^i \psi_c(x) | J/\Psi \rangle \\
& + \dots
\end{aligned} \tag{19}$$

where  $\dots$  stands for the contributions from the terms at  $v^2$  in  $\Gamma(q, P)$ . The  $x$ -dependence in the NRQCD matrix element can be expanded. This expansion will lead to a tower of operators with  $n$ -derivative with  $x^0$ . Similar cases in quarkonium productions are considered [6], where one can add the effect due to these operators to cross-sections at the leading order of  $v$  and it results in that the hadronic phase-space is recovered. Here we start with the hadronic phase-space and try to approximate it with the partonic-phase space. After the expansion we obtain the  $S$ -matrix element up to order of  $v^2$ :

$$\begin{aligned}
\langle \ell^+(k_1), \ell^-(k_2) | S | J/\Psi(P) \rangle = & ie^2 Q_c \bar{u}(k_2) \gamma^i v(k_2) \\
& \cdot \{ (2\pi)^4 \delta^4(p_0 - k_1 - k_2) \left( \frac{1}{4m_c^2} \langle 0 | \chi_c^\dagger \sigma^i \psi_c | J/\Psi \rangle \right. \right. \\
& + \frac{7}{24m_c^4} \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma^i \psi_c | J/\Psi \rangle \Big) \\
& + \frac{1}{4m_c^3} \frac{\partial}{\partial k_1^0} (2\pi)^4 \delta^4(p_0 - k_1 - k_2) \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma^i \psi_c | J/\Psi \rangle \Big\}
\end{aligned} \tag{20}$$

With the above  $S$ -matrix element we obtain the decay width

$$\begin{aligned}\Gamma(J/\Psi \rightarrow \ell^+ \ell^-) &= \alpha^2 Q_c^2 \frac{2\pi}{3} \frac{1}{m_c^2} [|\langle 0 | \chi_c^\dagger \sigma \psi_c | J/\Psi \rangle|^2 \\ &+ \frac{2}{3m_c^2} (\langle J/\Psi | \psi_c^\dagger \sigma \chi_c | 0 \rangle \cdot \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma \psi_c | J/\Psi \rangle + h.c.)] + \mathcal{O}(v^4)\end{aligned}\quad (21)$$

where the spin average for  $J/\Psi$  is implicit in the products of the matrix elements. This result agrees with that derived by the matching procedure in [1]. The relativistic correction is represented by the product of matrix elements:

$$a = \langle J/\Psi | \psi_c^\dagger \sigma \chi_c | 0 \rangle \cdot \langle 0 | (\mathbf{D}^2 \chi_c)^\dagger \sigma \psi_c | J/\Psi \rangle + h.c.. \quad (22)$$

The decay width depends on  $M_{J/\Psi}$  only implicitly through the matrix elements, while originally there is an explicit dependence on  $M_{J/\Psi}$  in the term detailed in Eq.(19). However this term can be treated by the direct expansion of the  $\delta$ -function

$$\begin{aligned}\delta(M_{J/\Psi} - k_1^0 - k_2^0) &= \delta(2m_c - k_1^0 - k_2^0) - \Delta M_{J/\Psi} \frac{\partial}{\partial k_1^0} \delta(2m_c - k_1^0 - k_2^0) + \mathcal{O}(v^4) \\ \Delta M_{J/\Psi} &= M_{J/\Psi} - 2m_c\end{aligned}\quad (23)$$

where  $\Delta M_{J/\Psi}$  is at order of  $v^2$  as shown before. With this expansion and  $\Gamma(q, P)$  given in Eq.(18) the decay width is

$$\begin{aligned}\Gamma(J/\Psi \rightarrow \ell^+ \ell^-) &= \alpha^2 Q_c^2 \frac{2\pi}{3} \frac{1}{m_c^2} [|\langle 0 | \chi_c^\dagger \sigma \psi_c | J/\Psi \rangle|^2 (1 + \frac{\Delta M_{J/\Psi}}{m_c}) \\ &+ \frac{7}{6m_c^2} a] + \mathcal{O}(v^4).\end{aligned}\quad (24)$$

Both results should be equivalent up to order of  $v^2$ , one can hence find a relation between  $a$  and  $\Delta_{J/\Psi}$  and rewrites the decay width as

$$\Gamma(J/\Psi \rightarrow \ell^+ \ell^-) = \alpha^2 Q_c^2 \frac{2\pi}{3} \frac{1}{m_c^2} [|\langle 0 | \chi_c^\dagger \sigma \psi_c | J/\Psi \rangle|^2 (1 - \frac{4\Delta M_{J/\Psi}}{3m_c}) + \mathcal{O}(v^4)]. \quad (25)$$

As discussed before, the quantity  $M_{J/\Psi}$  is negative, hence the relativistic correction seems to enhance the decay width. This is in contrast to the model calculation where one uses a wave-function computed in nonrelativistic potential models. With the wave-function the parameter  $a$  can be estimated as

$$a = \frac{1}{2\pi} R^*(r) \nabla^2 R(r)|_{r=0} + h.c. \quad (26)$$

where  $R(r)$  is the radial wave function. From above  $a$  is negative, the decay width therefor is reduced. But as discussed in [1], this parameter estimated in this way is not well defined because it is divergent if the potential is Coulombic at short distances. Certain subtraction is need to make it finite. It is unclear how to do the subtraction in a model calculation to make the estimation meaningful for the renormalized matrix element in  $a$ . In model calculations certain regularization is used to obtain a finite relativistic correction, like in [7].

In rewriting the decay width we used the relation between  $a$  and  $\Delta M_{J/\Psi}$ . It should be kept in mind that this relation is not proved in general and it applies only in the detailed cases considered here. Similar analysis can be done for  $\eta_c \rightarrow \gamma\gamma$  and the result is that the kinematic effect does not appear at the next-to-leading order of  $v$ .

The other process to be considered in this work is  $J/\Psi \rightarrow$  light hadrons. It should be noted that for processes involving strong interaction NRQCD factorization is in general performed for squares of amplitudes. However, the vacuum saturation is a good approximation in NRQCD, this enables us to work at the level of amplitude. With the vacuum saturation and at leading order of  $\alpha_s$  the process to be considered is  $J/\Psi \rightarrow 3$  Gluons. The  $S$ -matrix element can be written:

$$\langle 3G|S|J/\Psi(P)\rangle = i(2\pi)^4 \delta^4(P - k_1 - k_2 - k_3) \int \frac{d^4q}{(2\pi)^4} \text{Tr}\{A(k_1, k_2, k_3, q)\Gamma^h(q, P)\}, \quad (27)$$

$$\Gamma_{ij}^h(q) = \int d^4x e^{iq \cdot x} \langle 0|\bar{c}_j(x)c_i(0)|J/\Psi\rangle, \quad (28)$$

$$\begin{aligned} A(k_1, k_2, k_3, q) = & -ig_s^3 T^{a_1} T^{a_2} T^{a_3} \frac{1}{(k_1 - q)^2 - m_c^2} \frac{1}{(k_1 + k_2 - q)^2 - m_c^2} \\ & \cdot \gamma \cdot \varepsilon_1^{a_1} (\gamma \cdot (k_1 - q) + m_c) \gamma \cdot \varepsilon_2^{a_2} (\gamma \cdot (k_1 + k_2 - q) + m_c) \gamma \cdot \varepsilon_3^{a_3} \\ & + \text{Permutation of } 1, 2, 3. \end{aligned} \quad (29)$$

The indices 1,2,3 are labels for the 3 gluons. To calculate the decay width up to  $v^2$  we match the nonperturbative object  $\Gamma^h(q)$  into NRQCD matrix element and only keep the contributions up to order of  $v^2$  and take the contribution at  $v^2$  from the  $\delta$ -function for the energy conservation into account. In the calculation of the effect at  $v^2$  one should use physical spin-sums for gluons and be careful with the integration over the 3-body phase-space. The detailed calculation is tedious but straightforward. We obtain:

$$\begin{aligned} \Gamma(J/\Psi \rightarrow \text{light hadrons}) = & \frac{20}{243m_c^2} \alpha_s^3 \{ |\langle 0|\chi_c^\dagger \sigma \psi_c|J/\Psi\rangle|^2 (\pi^2 - 9) \\ & + (\frac{101}{192}\pi^2 - \frac{7}{2}) \cdot \frac{1}{m_c^2} a \} + \mathcal{O}(v^4). \end{aligned} \quad (30)$$

Similarly as for the leptonic decay we can directly expand the  $\delta$ -function for the energy conservation and obtain the decay width as

$$\begin{aligned} \Gamma(J/\Psi \rightarrow \text{light hadrons}) = & \frac{20}{243m_c^2} \alpha_s^3 \{ |\langle 0|\chi_c^\dagger \sigma \psi_c|J/\Psi\rangle|^2 (\pi^2 - 9 + \frac{352 - 33\pi^2}{32} \frac{\Delta M_{J/\Psi}}{m_c}) \\ & + \frac{192 + \pi^2}{96} \cdot \frac{1}{m_c^2} a \} + \mathcal{O}(v^4). \end{aligned} \quad (31)$$

Both widths are correct up to order of  $v^2$ , we can hence rewrite the width as:

$$\begin{aligned} \Gamma(J/\Psi \rightarrow \text{light hadrons}) = & \frac{20}{243m_c^2} \alpha_s^3 \{ |\langle 0|\chi_c^\dagger \sigma \psi_c|J/\Psi\rangle|^2 [\pi^2 - 9 - (\frac{101}{96}\pi^2 - 7) \frac{\Delta M_{J/\Psi}}{m_c}] \\ & + \mathcal{O}(v^4). \end{aligned} \quad (32)$$

If we take  $m_c = 1.65\text{GeV}$  as before, the width for the decay into leptons and into light hadrons is enhanced at the level of 16% and of 48% respectively.. The hadronic width receives

a substantial correction. From Eq.(32) and Eq.(25) one can also obtain the corresponding widths for  $\Upsilon$ . The same level of the enhancement is also found for  $\Upsilon \rightarrow \ell^+\ell^-$  and for  $\Upsilon \rightarrow$  light hadrons. To compare with experimental data we build the ratio:

$$\begin{aligned} r_b &= \frac{\Gamma(\Upsilon \rightarrow \text{light hadrons})}{\Gamma(\Upsilon \rightarrow \ell^+\ell^-)} \\ &= 5040.0\alpha_s^3(m_b)(1.276 - 0.081\alpha_s(m_b)) \end{aligned} \quad (33)$$

where we have taken  $M_\Upsilon = 9.460\text{GeV}$  and the pole mass  $m_b = 5.0\text{GeV}$ . We have also added the one-loop QCD correction in each decay width [8,9]. In this ratio the one-loop correction is small, while the correction in each decay width is large. With the experimental value  $r_b = 37.04$  we can estimate  $\alpha_s(m_b)$  as

$$\alpha_s(m_b) = 0.18. \quad (34)$$

which is close to the value of  $\alpha_s$  by running down from the scale  $\mu = M_Z$ . However, the relativistic correction is not significant in the determination of  $\alpha_s$ . Similar estimation can be done for  $J/\Psi$ , the determined  $\alpha_s(m_c = 1.6\text{GeV})$  is 0.195. It is smaller than the expected. This indicates that effects in higher orders of  $v$  may be still significant. It also should be noted that the pole mass  $m_c$  is not known precisely as  $m_b$ .

To summarize: In this work we tried to define the difference between a quarkonium mass and the twice of the heavy quark mass in terms of NRQCD matrix elements. We separated the effect introduced by the difference in relativistic corrections for several decays of quarkonium and find that the relativistic correction is determined by the difference. With the determination decay widths considered here are enhanced by the relativistic correction.

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